Einstein's Theory of Relativity, PHY 27
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Summary of Concepts

Space-time
Parallel transport
Definition of curvature
Some examples of curved space

Matter determines the "rest-frame" of the Universe. The rest frame is that for which the spectrum of the background thermal radiation is independent of direction. The Earth and all of the matter within the Universe is in relative motion, whereby the spectrum of the thermal background depends on direction due to Doppler shifting caused by this relative motion. The absolute rest frame is that for which the spectrum is the same in all directions.

Space-time is given by $d\tau^2 = g_{\mu\upsilon} \cdot dx^\mu \cdot dx^\upsilon$

In special relativity the metric $\eta_{\mu\upsilon}$ is constant.

$$\eta_{\mu\upsilon} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$d\tau^2 = dt^2 - \frac{dx^2}{c^2} - \frac{dy^2}{c^2} - \frac{dz^2}{c^2} = t^2 - dx^2 - dy^2 - dz^2 \,\big|_{c=1}$$

Time is t and proper time is τ. If the motion is slow with respect to c, then $d\tau^2 = dt^2$. For flat space we can always find a coordinate system for which the metric tensor is constant and its derivative is zero.

Assume $\dfrac{\partial g_{mn}(x)}{\partial x^r} \neq 0$.

Define new coordinate system $y^m = x^m + C^m_{rs} \cdot x^r \cdot x^s$

C is symmetric because $x^r x^s = x^s x^r$

There are *d(d+1)/2* independent *C*'s when the number of dimensions is *d*. Transform from *y* to *x* to get the metric.

$$\frac{\partial g_{mn}(y)}{\partial y^s} = 0$$

One can always solve this equation to find the coordinates.

$$\Gamma^r_{mn} = \frac{1}{2} \cdot g^{rs} \cdot \left\{ \frac{\partial g_{sm}}{\partial x^n} + \frac{\partial g_{sn}}{\partial x^m} - \frac{\partial g_{mn}}{\partial x^s} \right\}$$

$$\therefore \nabla g = \frac{\partial g}{\partial x} + \Gamma \cdot g + \Gamma \cdot g = 0$$

Because we could do this everywhere (but with different C's), then $\nabla g = 0$ everywhere, and the definitions above are consistent. Curvature is a mathematical obstruction that prevents us from finding coordinates that are flat.

Parallel transport is the process of moving a vector along a space curve without changing its direction. This is easy in flat space. How about in curved space?

If the covariant derivative of the tangent vector of a curve is zero, then the curve is a geodesic. Let V be a vector on a space curve.

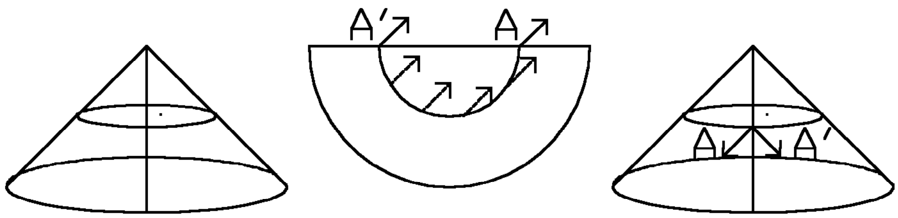
$$\frac{dV^m}{ds} + \Gamma^m_{np} \cdot V^n \cdot \frac{dx^p}{ds} \quad \text{where} \quad \frac{dx^p}{ds} \text{ is a tangent vector.}$$

$$.V^m \rightarrow \frac{dx^m}{ds}. \qquad \frac{d^2x^m}{ds^2} + \Gamma^m_{np} \cdot \frac{dx^n}{ds} \cdot \frac{dx^p}{ds} = 0$$

Given an arbitrary vector on a curve, if the covariant derivative is zero, then the vector has been parallel-transported along the curve. One can always do this. The following equation shows how to change components of *V* to achieve parallel transport. One can achieve parallel transport on curves that are not geodesics.

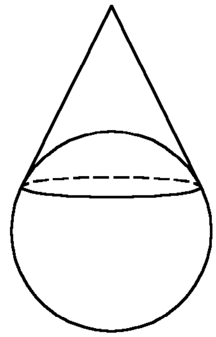
$$dV^m = -\Gamma^m_{np} \cdot V^n \cdot dx^p$$

Parallel transport of a vector around a closed curve that does not contain points that are curved will result in it returning to its point of departure with no angular difference in direction (no deflection) with respect to its original direction. If there is a point of curved space within the area enclosed by the curve, there will be a deflection of the vector during parallel transport.



Consider the cone above as an example of curved space. The only point of a cone that is curved is the apex, which is also a point of singularity. Consider a closed curve that encircles the apex of the cone. Cut the cone, unroll it and parallel-transport a vector (*A*) from one edge of the cut to the other (*A'*). Reassemble the cone to find that *A'* does not point in the direction of *A*. The deflection angle as a result of parallel transport will not be zero if the path encloses a point of curved space. For a cone the deflection angle depends on the angle of the apex.

Now consider a sphere with a circular path (not a great circle, which would be a geodesic) as shown below. The deflection of a parallel-transported vector will be the same as that for the cone constructed on the sphere as shown. The deficit is proportional to the area within the closed path.



The deficit angle is related to the curvature as follows.
$\delta\theta = R \cdot \delta a$

The sign of the curvature (R) is defined as positive if the deficit angle is positive when parallel transport is counter-clockwise along the closed curve, i.e., if the sign of $\delta\theta$ is the same as that of δa. The curvature of the cone is everywhere zero except at the apex where it is positive infinity. The curvatures of the points of the outer surface of a torus are positive, while the curvatures of the points of the inner surface are negative. There are as many points of positive curvature as those of negative curvature of the surface of a torus. The curvature of all points of a spherical surface is positive.

End of lecture # 6