Einstein's Theory of Relativity, PHY 27
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Session 10, November 23, 2008

Summary of Concepts

Possibilities of spatial dimensions other than three
Cosmological Constant
Wave Equation for curved space
Equivalence
Non-zero cosmological constant implies empty space is curved

If there is no energy density in space, then the space is flat. Place a particle in space, and at that point space becomes curved in the shape of a cone. What if our space comprised other than three spatial dimensions? With one spatial dimension, a mass as small as the Planck mass would generate enough curvature to close up the space (a cone with zero apex angle and infinite time extent). As the number of dimensions increases, so does the inverse law. If the spatial dimensions were four, then an inverse cube law would obtain and atoms would not be stable.

$$R_{\mu\upsilon} - \frac{1}{2} \cdot g_{\mu\upsilon} \cdot R = 8 \cdot \pi \cdot G \cdot T_{\mu\upsilon}$$

Require $\nabla_{\mu} T^{\mu\upsilon} = 0$ for continuity. Then for each component

$$\left\{ \begin{array}{l} \nabla_{\mu} G^{\mu\upsilon} = 0 \\ \nabla_{\mu} g^{\mu\upsilon} = 0 \end{array} \right\}$$ for covariant conservation.

The covariant derivatives of the Metric tensor are always zero.

$G^{\mu\upsilon} + \Lambda \cdot g^{\mu\upsilon} = k \cdot T^{\mu\upsilon}$ where $k = 8 \cdot \pi \cdot G$ and Λ is the

Cosmological constant. The Cosmological constant adds an attraction or repulsion depending on its sign. It was assumed to be zero by Einstein. If the constant is small, then its repulsive force will be negligible until very far away from the observer.

Let us review the equivalence of General Relativity with the Newtonian theory.

$$R_{\mu\upsilon} - \frac{1}{2} \cdot g_{\mu\upsilon} = k \cdot T_{\mu\upsilon} \qquad \nabla^2 g_{00} = k \cdot \rho$$

Einstein assumed that ρ is the energy density and ∇^2_{00} is the Newtonian gravitation potential. Special Relativity applies in a small patch of space far from any energy density. These equations are taken to tensors in order to transition from Special relativity to General relativity.

A particle in an acceleration field (gravitational field) moves along geodesic curves. How about energy density without mass, i.e. motion of a wave in space? Consider the simple wave equation as follows.

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \cdot \frac{\partial^2 \phi}{\partial x^2}$$

The solution to this equation is a wave (or super position of waves) moving to the left or right with a velocity of *v*. Now consider the equation in three dimensions with *v* = 1.

$$\eta^{\mu\upsilon} \cdot \frac{\partial^2 \phi}{\partial x^\mu \partial x^\upsilon} = 0 \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \eta_{00} = time,\ \eta_{11}, \eta_{22}, \eta_{33} = space$$

But this is not a tensor equation because the derivatives are not covariant derivatives. The proper tensor wave equation for curved space is as follows.

$$\nabla_{\mu} g^{\mu\upsilon} \cdot \frac{\partial\phi}{\partial x^{\upsilon}} = \frac{\partial}{\partial x^{\mu}} g^{\mu\upsilon} \cdot \frac{\partial\phi}{\partial x^{\upsilon}} + \Gamma^{\mu}_{\alpha} g^{\alpha\beta} \cdot \frac{\partial\phi}{\partial x^{\beta}} = 0$$

In this equation Γ is a kind of gravitational force that bends the wave, such that two beams of light passing each other would be deflected towards each other but deflection is very small because

$$G \cdot \frac{m_1 \cdot m_2}{r^2} = G \cdot \frac{E_1 / c^2 \cdot E_2 / c^2}{r^2}.$$

If $\frac{\partial\phi}{\partial x^{\mu}} = \nabla_{\mu}\phi$, then $\nabla_{\upsilon} g^{\mu\upsilon} \nabla_{\mu}\phi = 0$

Start with Special relativity.

$$\eta^{\mu\upsilon} \cdot \partial_{\mu}\partial_{\upsilon}\phi = \partial_{\mu}\partial^{\mu}\phi = 0$$

But we need ϕ to have two indices, say $T_{\mu\upsilon}$.

if $T_{\mu\rho} = \partial_{\mu}\phi \cdot \partial_{\upsilon}\phi + C \cdot \eta_{\mu\upsilon} \cdot \partial_{\sigma}\phi \cdot \partial^{\sigma}\phi$

what value of C satisfies the continuity equation?

$$\partial^{\mu} T_{\mu\upsilon} = \partial^{\mu}\left(\partial_{\mu}\phi \cdot \partial_{\upsilon}\phi\right) + \partial_{\mu}\phi \cdot \partial^{\mu}\partial_{\upsilon}\phi + \partial_{\upsilon}\partial_{\sigma}\phi \cdot \partial^{\sigma}\phi + \partial_{\mu}\partial_{\upsilon}\phi \cdot \partial^{\mu}\phi$$

This implies that C = -1/2.

$$T_{\mu\upsilon} = \partial_{\mu}\phi \cdot \partial_{\upsilon}\phi - \frac{1}{2} \cdot \eta_{\mu\upsilon} \cdot \partial_{\sigma}\phi \cdot \partial^{\sigma}\phi$$

The energy density is given by

$$T_{00} = \frac{1}{2} \cdot \left[\dot{\phi}^2 + \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right]$$

Where the first term in the parenthesis (time) is the potential energy of the wave and the other three terms (space) are the kinetic energy of the wave. However, there is an ambiguity as we can multiply this by a constant (say *k*). Now take this to general relativity. Energy pressure and stress are part of the gravitational field.

$$R_{\mu\upsilon} - \frac{1}{2} \cdot g_{\mu\upsilon} \cdot R = k \cdot T_{\mu\upsilon}$$

Therefore, energy and momentum determine how to bend space, while the bent space determines how the wave will propagate. Put in the cosmological constant λ.

$$R_{\mu\upsilon} - \frac{1}{2} \cdot g_{\mu\upsilon} \cdot R + \lambda \cdot g_{\mu\lambda} = k \cdot T_{\mu\upsilon}$$

What is Newtonian analogy?

$$\nabla^2 \phi + \lambda = 4 \cdot \pi \cdot G \cdot \rho$$

What if $\nabla^2 \phi = \lambda$ (uniform mass density)?

$$\phi = \frac{\lambda}{6} \cdot \left(x^2 + y^2 + z^2 \right)$$

Differentiate to get force.

$$\frac{\partial \phi}{\partial x} = \frac{\lambda}{3} \cdot x$$

Everything is pushed away from observer Or, if we put λ on the other side of the equation, then it is vacuum energy. So, either a uniform mass is pushing everything outward or a vacuum energy is doing it. But λ is very small, so such repulsion is detectable only for very large distances away from observer. Einstein chose a value for λ such that expansion matched attraction – but this was a mistake because such a balance is not stable.

Gravitational waves generate shear (non-zero off axis) terms in T. Eigen values of T have a significant meaning.

If $\lambda \neq 0$, then $R_{\mu\upsilon} - \frac{1}{2} \cdot g_{\mu\upsilon} \cdot R = \lambda \cdot g_{\mu\upsilon}$

$R - 2 \cdot R = 4 \cdot \lambda$ and $R = -4 \cdot \lambda$

At infinity (and with no matter) the curvature of space is not zero. Therefore, with non-zero cosmological constant space is uniformly curved. Note that flat space and flat space-time are different.

End Lecture # 10