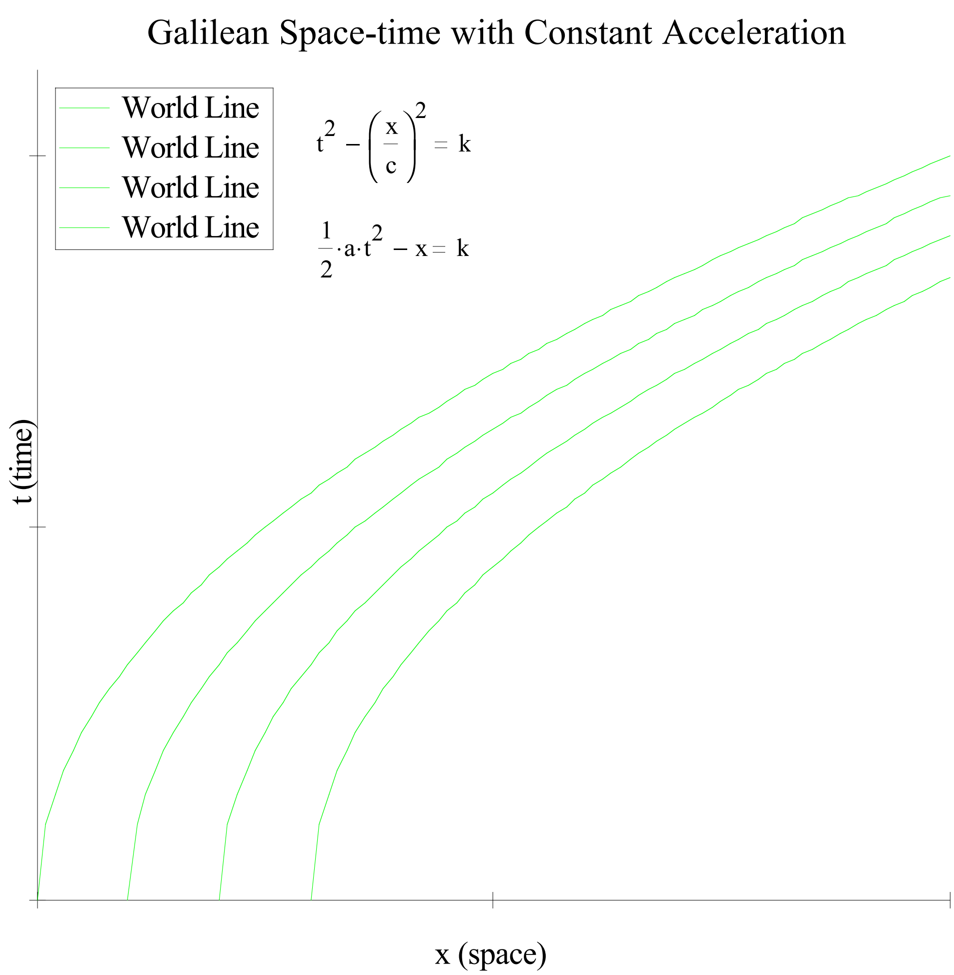
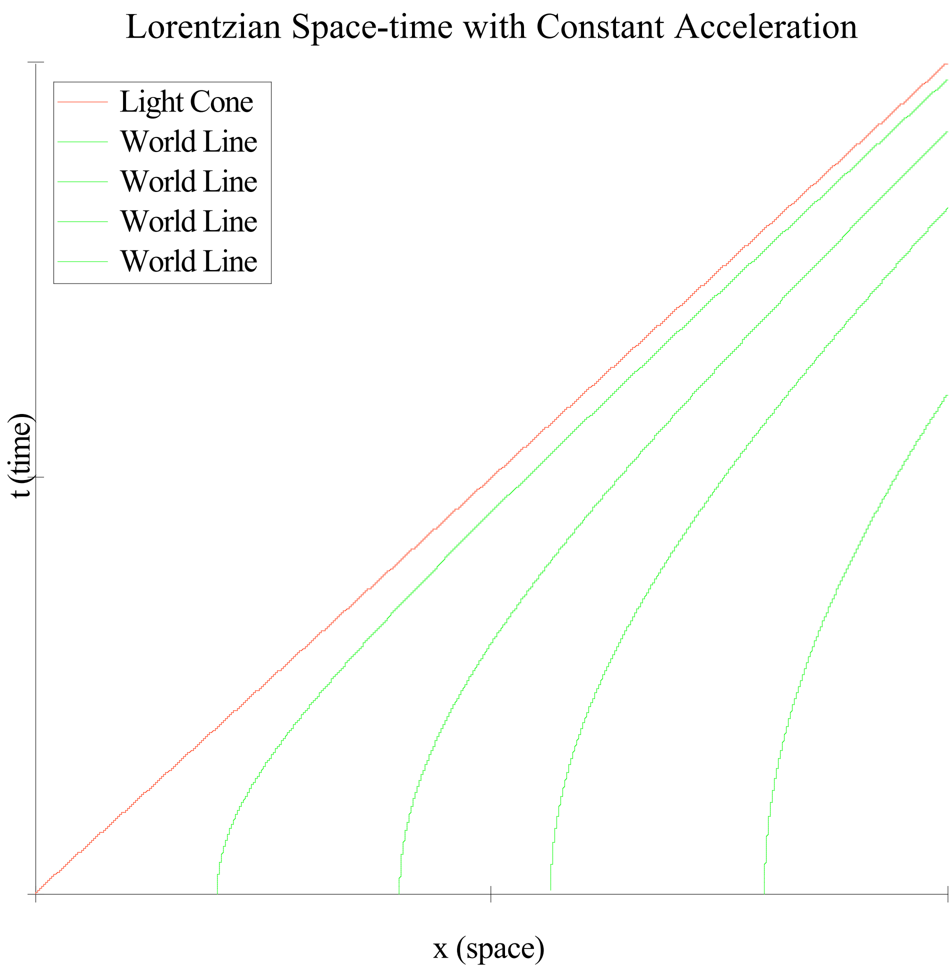
Einstein's Theory of Relativity, PHY 27
Professor Susskind
Session 12, December 8, 2008

Summary of Concepts

Space-Time Diagrams





The Schwarzschild solution is

$$d\tau^2 = \left(1 - \frac{2 \cdot M}{r}\right) \cdot dt^2 - \frac{1}{1 - \frac{2 \cdot M}{r}} \cdot dr^2 - r^2 \cdot d\Omega^2 \Big|_{c=1}$$

where $d\Omega^2 = d\phi^2 + \sin^2\phi \cdot d\theta^2$

There are two singularities in this equation, one of which is not a problem and the other of which is a problem. As observer approaches the event horizon (i.e., $r = 2 \cdot M \cdot G$), his clock ticks ever slower until at the horizon proper time and ordinary time become disconnected. But this singularity is just a consequence of our choice of coordinates, so it is not a real problem. The second is that the coefficient of dr^2 becomes infinite. Also, at r=0, the coefficient of dt^2 blows up with negative sign and dr^2 goes to -zero. The signs of dt^2 and dr^2 now change at the horizon. Something funny is happening at the even horizon.

Now invent another variable. $\rho \to R,\ d\rho \to \dfrac{dR}{d\rho}$

$$d\tau^2 = R \cdot d\omega^2 - \frac{1}{4 \cdot \rho^2} \cdot dR^2 = R \cdot d\omega^2 - \frac{1}{4 \cdot \rho^2} \cdot dR^2$$

$\omega \to 0 @ R = 0 \,\&\, dR^2 \to \infty$ This is not a problem because it is caused and can be resolved by a selection of coordinates.

When R goes from + to -, then space and time are swapped for each other. And $\rho^2 < 0$ is a location above the light cone.

The Schwarzchild radius is the horizon $r = 2 \cdot M \cdot G$

$$d\tau^2 = \left(\frac{r - 2 \cdot M \cdot G}{r} \right) \cdot dt^2 - \left(\frac{r}{r - 2 \cdot M \cdot G} \right) \cdot dr^2$$

From outside move close to the event horizon.

$$\approx \left(\frac{r - 2 \cdot M \cdot G}{2 \cdot M \cdot G} \right) \cdot dz^2 - \left(\frac{2 \cdot M \cdot G}{r - 2 \cdot M \cdot G} \right) \cdot dr^2 - (2 \cdot M \cdot G)^2 \cdot d\Omega^2$$

The last term in the equation above is a sphere of radius 2MG but approximately a flat (tangent) plane.

$$\frac{r-2\cdot M\cdot G}{2\cdot M\cdot G}\cdot dt^2-\frac{2\cdot M\cdot G}{r-2\cdot M\cdot G}\cdot dr^2-dy^2-dz^2$$

Change coordinates again (until see something that is familiar). Make the metric look like polar coordinates ($d\tau^2=\rho^2\cdot d\omega^2-d\rho^2$).

$d\tau^2=-d\rho^2$ (proper distance on space-time diagram)

What is the proper distance from event horizon to r? It is determined by the *dr* term (time remains the same).

$$ds^2=\frac{2\cdot M\cdot G}{r-2\cdot M\cdot G}\cdot dr^2 \qquad ds=\pm\sqrt{\frac{2\cdot M\cdot G}{r-2\cdot M\cdot G}}dr$$

Let ρ be the proper distance from the horizon to a small distance outside. Rewrite the metric with a new coordinate ρ (we will reuse ρ).

$$\rho=\sqrt{8\cdot M\cdot G}\sqrt{r-2\cdot M\cdot G} \qquad \frac{\rho^2}{8\cdot M\cdot G}=r-2\cdot M\cdot G$$

$$ds^2=\frac{\rho^2}{8\cdot M\cdot G\cdot 2\cdot M\cdot G}\cdot dt^2-d\rho^2-dz^2-dy^2$$

Let $\omega=\frac{t}{4\cdot M\cdot G}$

The near horizon solution is

$$\rho^2 \cdot \left(\frac{dt}{4 \cdot M \cdot G} \right)^2 - d\rho^2 - dz^2 - dy^2$$

Therefore, the near-horizon solution is close to the variable difference of observers of accelerated space-time.

Far away from horizon $d\tau^2 \approx dt^2 - r^2 \cdot d\Omega^2$, i.e., flat space.

Horizon grows as mass gets close (but nothing passes through the horizon!!! As mass merges with the hole). The relationship between two observers near the horizon is the same as the relationship between observers in accelerated space time as one approaches the light cone.

At r =0 there is a real singularity. Anything moving along the horizon is moving faster than the speed of light, so it cannot happen. Now consider r small, then + becomes – and an interchange between space and time has taken place. Inside the hole space is time and not a place.

What is the density required to form a black hole?

$$\frac{2 \cdot M \cdot R}{R^2} = \frac{R}{R^2} \qquad \frac{1}{R^2} = \frac{1}{(2 \cdot M \cdot R)^2}$$

However, even classically cannot form a singularity by collapsing mass because of the angular momentum of the mass falling inward. Once the mass is smaller than the Schwarzchild radius, you have a black hole.

end lecture #12