Einstein's Theory of Relativity, PHY 27
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Summary of Concepts

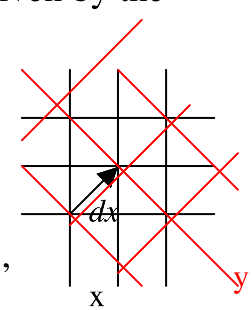
Transformation of Coordinates
Contravariant and Covariant Transformations
Metric Tensors
Flat and Curved Spaces

Transformation of Coordinates

The infinitesimal vector between a point and its neighbor in a coordinate system *x* of dimension *d* is given by the infinitesimal differences of its vector components. We will indicate indices by superscripts.

$d\vec{x} = (dx^1, dx^2 ..., dx^d)$, and $dx = \sqrt{(dx^1)^2 + (dx^2)^2 + ... + (dx^3)^2}$.

The change in the value of a function $d\phi(x)$ in going an infinitesimal distance from one point to its neighbor in coordinate system *x* is given by the following rule.



$$d\varphi = \frac{\partial \phi}{\partial x^n} \cdot dx^n \qquad \text{where} \qquad \frac{\partial \varphi}{\partial x^n} \cdot dx^n \equiv \sum_{n=1}^{d} \frac{\partial \varphi}{\partial x^n} \cdot dx^n$$

In order to simplify the equations, we will use Einstein's notation , wherein the summation sign is omitted, and the index *n* will be understood to be a dummy variable.

Given a second coordinate system $y(x)$, we may use the following rule to find the value of $d\phi(y)$ at the same point but in terms of the *y* system.

$$d\varphi = \frac{\partial \varphi}{\partial y^n} \cdot dy^n \qquad \text{wherein} \qquad \frac{\partial \varphi}{\partial y^n} = \frac{\partial \varphi}{\partial x^m} \cdot \frac{\partial x^m}{\partial y^n}$$ and m is a dummy variable of summation.

Tensors

A Tensor is a multi-component object that transforms vectors and functions from one coordinate system to another in the same space. Think of a vector as a physical stick. Tensors are classified by rank. A Tensor of rank 0 is a scalar. A Tensor of rank 1 is a vector. A Tensor of rank 2 is a matrix also called a Dyadic. Given an infinitesimal vector in system *x*,

$d\vec{x} = (dx^1, dx^2 .. dx^d)$ and the same vector in system *y*, $d\vec{y} = (dy^1, dy^2 .. dy^d)$.

We can convert the *n*th coordinate from the *x* system to the *y* system as follows.

$dy^n = \frac{\partial y^n}{\partial x^m} \cdot dx^m$ wherein m is a dummy index of summation.

Covariance and contravariance refer to how coordinates change under a change of basis of the same space. Covariant components have lower indices, while contravariant components have upper indices. A covariance transformation is the inverse of a contravariant trandformation. An essential feature is that the dot-product of a contravariant vector with a covariant tensor is a scalar.

A vector $V_{(y)}$ relative to *y* is transformed to a vector $V_{(x)}$ relative to *x* by a contravariant transformation as follows.

$V^n_{(y)} = \frac{\partial y^n}{\partial x^m} \cdot V^m_{(x)}$, **Contravariant transformation** wherein $V^n_{(y)}$ is the *n*th component of the vector *V* relative to *y*.

A tensor of rank 2, which has d^2 components, may be written in terms of two vectors as follows.

$T = AB$ or $T^{mn} = A^m \cdot B^n$ How does it transform?

$$T^{mn}_{(y)} = A^m_{(y)} \cdot B^n_{(y)} = \frac{\partial y^m}{\partial x^r} \cdot A^r_{(x)} \cdot \frac{\partial y^n}{\partial x^s} \cdot B^s_{(y)} = \frac{\partial y^m}{\partial x^r} \cdot \frac{\partial y^n}{\partial x^s} \cdot A^r_{(x)} \cdot B^s_{(x)} = \frac{\partial y^m}{\partial x^r} \cdot \frac{\partial y^n}{\partial x^s} \cdot T^{rs}_{(x)}$$

Each time there is a transfer, an additional $\frac{\partial y^m}{\partial x^r}$ is added. Find the pattern.

The partial derivatives increase by one for each additional rank of the tensor. The indices of contravariant tensors are written as superscripts. The indices of covariant tensors are written as subscripts. An object A (functional) is transformed covariantly as follows.

$$A_n^{(y)} = \frac{\partial x^n}{\partial y^m} A_m^{(x)} \quad \textbf{Covariant transformation}$$

Equations should be consistent with respect to the position (subscript or superscript) of the summation indices. Summation indices of the transformation are always all upstairs or downstairs as follows (i.e., in the partials and subscripts of the object after the partials).

$$T_{mn}^{(y)} = \frac{\partial x^m}{\partial y^r} \cdot \frac{\partial x^n}{\partial y^s} \cdot T_{rs}^{(x)}$$

Metric Tensors

Flat space has a flat coordinate system. An example of flat space with flat coordinates is Euclidean space with Cartesian coordinates. Curved space cannot have flat coordinates (mathematically obstructed). A Metric Tensor generalizes many of the familiar properties of the inner product of vectors in Euclidean space. It can be used to compute the length of a vector, the angle between vectors or area contained within vectors. Consider the length (*ds*) of an infinitesimal vector $d\vec{x}$.

$$(ds)^2 = \delta_{mn} \cdot dx^n \cdot dx^m \quad , \quad \text{where} \quad \delta_{mn} \equiv \left\{ \begin{array}{c} 1 \Leftrightarrow m = n \\ 0 \Leftrightarrow m \neq n \end{array} \right\} = \text{ Kronecker Delta.}$$

We rewrite this length in curvilinear coordinates. We must always sum over one superscript and one subscript.

$$(ds)^2 = \delta_{mn} \cdot \frac{\partial x^m}{\partial y^r} \cdot \frac{\partial x^n}{\partial y^s} \cdot dy^s \cdot dy^r = g_{rs}(y) \cdot dy^s \cdot dy^r$$

$$g_{rs}(y) = \delta_{mn} \cdot \frac{\partial x^m}{\partial y^r} \cdot \frac{\partial x^n}{\partial y^s}$$

g is a **metric tensor** that transforms covariantly. In Cartesian coordinates δ_{mn} is a metric tensor.

$$(ds)^2 = (dx)^2 + (dy)^2 = g_{mn} \cdot dx^m \cdot dx^n \qquad g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta$$

Flat and Curved Space

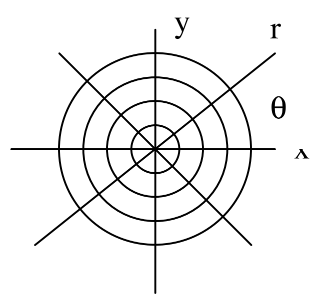
Suppose we do not assume Cartesian coordinates.

$(ds)^2 = g_{mn}(x) \cdot dx^m \cdot dx^n$, wherein $g_{mn}(x)$ is a metric tensor of rank 2.

Given $g_{mn}(x)$ exists, does there exist a coordinate transformation that will bring it back to flat coordinates, wherein it is not a function of the coordinates? If yes, then the space is flat. If no, then the space is curved. It is very difficult in general to determine that a coordinate system is not flat. Let us now consider Polar coordinates.

$$x = r \cdot \cos(\theta) \qquad y = r \cdot \sin(\theta)$$

$$ds^2 = dx^2 + dy^2 \qquad dx = \cos(\theta) \cdot dr - r \cdot \sin(\theta) \cdot d\theta$$



$$dy = \sin(\theta) \cdot dr + r \cdot \cos(\theta) \cdot d\theta$$

$$ds^2 = dr^2 + r^2 \cdot d\theta^2$$

$$g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad g_{rr} = 1,\ g_{\theta\theta} = r^2 \text{ and } g_{r\theta} = g_{\theta r} = 0.$$

Lines of constant r are circles and lines of constant $\theta$are radial lines. There are no cross-terms because the lines of each coordinate are perpendicular to the other. If the lines were not perpendicular (i.e., oblique coordinates), then there would be cross-terms, but the space could still be flat. This space is flat because these coordinates can be transformed to Cartesian coordinates.

The space comprising the surface of a sphere is an example of a space that is not flat. Let latitude = ϕ that is 0 at the South Pole and longitude = θ. Assume $r = 1$.

$$ds^2 = d\varphi^2 + \sin^2(\theta) \cdot d\theta^2$$

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix} \qquad g_{\phi\phi} = 1 \qquad g_{\phi\theta} = g_{\theta\phi} = 0 \qquad g_{\theta\theta} = \sin^2(\theta)$$

This expression looks similar to that of polar coordinates except for $g_{\theta\theta}$.

End of lecture #3