Einstein's Theory of Relativity, PHY 27
Professor Susskind
Session 4, October 13, 2008

Summary of Concepts

Contravariant and Covariant Transformations continued
Contraction of Indices
Tensor Algebra
Space-time

The bare minimum knowledge of mathematics required for General Relativity is still somewhat complicated, as we will now see. Given a vector $(\vec{V})$ with coordinate unit vectors $\hat{i}^1$ and $\hat{i}^2$

$$\vec{V} = C_1 \cdot \hat{i}^1 + C_2 \cdot \hat{i}^2$$

Then the unit vectors $(\hat{i})$ are contravariant, while the scalars (*C*) are covariant. It is the same with Cartesian coordinates, but it is not true in general (such as in the case of oblique coordinates, wherein lines of constant coordinate are not orthogonal).

$$V_m(y) = \frac{\partial x^n}{\partial y^m} \cdot V_n(x) \quad \text{covariant object}$$

$$W^m(y) = \frac{\partial y^m}{\partial x^n} \cdot W^n(x) \quad \text{contravariant object}$$

The indices for contravarient objects are subscripts, and the same subscript goes with the same coordinate in the equation. Consider more than one index of a covariant transformation.

$$T_{mr}^{(y)} = \frac{\partial y^n}{\partial x^m} \cdot \frac{\partial x^s}{\partial y^r} \cdot T_{ns}^{(x)}$$

Scalars have no indices left over after the summation. Consider a mixed (covariant and contravariant) tensor.

$$T^n_m(y) = \frac{\partial x^r}{\partial y^m} \cdot \frac{\partial y^n}{\partial x^s} \cdot T^s_r(x)$$

If the indices are the same, then the tensor is a scalar.

$$T^m_m = T^1_1 + T^2_2 + T^3_3 + \ldots$$

$$T^m_m(y) = \frac{\partial x^r}{\partial y^m} \cdot \frac{\partial y^m}{\partial x^s} \cdot T^s_r(x) = \delta^r_s \cdot T^s_r(x) = I^r_s \cdot T^s_r(x) = T^r_r(x)$$

The above tensor has no dependence on a coordinate system, so it is a scalar. Given a 6th rank tensor with 3 covariant indices and 3 contravariant indices.

$W^{mnra}_{pqm} = Q^{nra}_{pq}$ wherein index m is a **contracting** index

$V^m W_m = T^m_m$ is an inner product.

The following contraction of indices results in a scalar.

$$ds^2 = dx^m \cdot dx^n \cdot g_{mn}(x)$$

Is g a tensor? How does it transform? Rewrite it in y notation as follows.

$$dx^m = \frac{\partial x^m}{\partial y^r} \cdot dy^r$$

Plug this expression into ds^2 above to get the following result.

$$ds^2 = \frac{\partial x^m}{\partial y^r} \cdot \frac{\partial x^n}{\partial y^s} \cdot g_{mn}(x) \cdot dy^r \cdot dy^s = g_{rs}(y) \cdot dy^r \cdot dy^s$$

Therefore, g_{mn} is a metric tensor that is symmetric and has an inverse. The identity tensor is the Kronecker delta.

$$\delta^n_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiply to contract indices and get Kronecker delta.
$(g^{-1})^{mr} \cdot g_{rn} = \delta^m_n$ where r is a contracting index.

The inverse of a metric tensor is a contravariant tensor, and it is indicated by superscript indices.

If $g_{mn} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$, then $g^{-1}_{mn} = g^{mn} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$.

The vector $(\vec{V})$ can be written in terms of its coefficients (V_m) as follows.

$$\vec{V} = V_m \cdot \hat{i}_m$$

G is symmetric because multiplication of scalars is commutative.

$$g_{12} \cdot dx^1 \cdot dx^2 = g_{21} \cdot dx^2 \cdot dx^1$$

$g^{mr} g_{rn} = \delta^m_n$ prove it!

$V^m = g_{mn} \cdot V_n$ or $g^{mn} \cdot V^m = V_n$

Therefore, can go back and forth, which is called **lowering or raising indices.** One can lower or raise any index. The following are examples of raising indices.

$g^{mr} \cdot T_{rn} = T^m_n$ $dx_m = g_{mn} \cdot dx^n$

The following are three ways to write the scalar ds^2.

$$dx^m \cdot dx^n \cdot g_{mn} = dx^m \cdot dx_m = ds^2$$

$$g_{mn} \cdot dx^m \cdot dx^n = dx^m \cdot dx_m = dx^2$$

$$dx^m \cdot dx_m = dx_m \cdot dx_n \cdot g^{mn} = ds^2$$

In tensor algebra one can add tensors, but only if they are the same. If all of the components of a tensor in one coordinate system are zero, then they are zero in all coordinate systems. If two tensors are equal in one coordinate system, then they are equal in all coordinate systems.

Space-Time

Proper space-time between two points in space is given as follows.

not this $ds^2 = dt^2 + dx^2$ but this $d\tau^2 = dt^2 - \frac{dx^2}{c^2}$

Proper space-time is expressed as time. The speed of light (c) is a conversion factor to make units agree. Since it is a constant, it is often defined as 1 to simplify the equations. The Lorentz transformation preserves as invariant proper time.

$d\tau^2 = dt^2 - \frac{dx^2}{c^2} - \frac{dy^2}{c^2} - \frac{dz^2}{c^2}$ or $d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$

Space-time may be written as follows.

$x^\mu = (x^0, x^1, x^2, x^3)$ or $y^\mu = (y^0 ...)$

$d\tau^2 = g_{\mu\upsilon}(x) \cdot dx^\mu \cdot dx^\upsilon$ x can be any of the space-time coordinates

$$g_{\mu\upsilon} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a(t) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

a(t) is called **scale factor**

$dt^2 - a(t)^2 \cdot dx^2$ is the expanding universe. An analog for ordinary space of one length coordinate and one angular coordinate is the following.

$ds^2 = dr^2 + r^2 \cdot d\theta^2$ plane (flat space).

$ds^2 = dr^2 + r^2 \cdot \sin(\theta)^2 \cdot d\theta^2$ spherical surface (curved space).

$ds^2 = dr^2 + r^2 \cdot e^{2r} \cdot d\theta^2$ exponential horn in θ direction (curved space).

Suppose $ds^2 = dr^2 + r^2 \cdot f(\theta) \cdot d\theta^2$ with arbitrary *f*.

The *f* determines complexity, which is what a(t) does.

End Lecture 4.