Einstein's Theory of Relativity, PHY 27
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Summary of Concepts

Tensor Algebra Continued
Derivatives of Tensors
Christofel Symbols and Gamma
Geodesics

Let $\vec{V} = V^n \cdot \hat{i}_n$ then $V_n = g_{mn} \cdot V^m$ and $g^{mn} \cdot V_n = V^m$

$dx_m = g_{mn} \cdot dx^n \qquad ds^2 = g^{mn} \cdot dx_m \cdot dx_n$

Tensors of the same type add and transform the same way. If two tensors are equal in one coordinate system, then they are equal in all coordinates systems, whereby the laws of Physics apply to all. If a tensor equals zero, then all its components equal zero.

The (covariant) components of vectors are projections on the coordinate axes of the space. Consider curvilinear coordinates. While the direction of a constant vector field is the same everywhere in the space, the values of the components are in general not. In general the components of a vector depend on both the vector and its coordinates. Therefore, that the derivative of one component of a vector equals zero **<u>does not</u>** imply that the derivative of that component in another coordinate equals zero. If the Metric tensor is not independent of coordinate (i.e., constant), then components of the vector depend on its position in the space.

$$\frac{\partial V_m(x)}{\partial x^n} = 0 \not\Rightarrow \frac{\partial V_m(y)}{\partial y^n} = 0 \, .$$

Derivatives of tensor components do not transform as tensors. In order to differentiate tensors, one must do **covariant differentiation** (the term covariant differentiation is entirely different from the use of covariance in coordinate transformations).

$$T_{rs}(x) = \frac{\partial V_r(x)}{\partial x^s} \neq \frac{\partial V_r(y)}{\partial y^s}$$

$$T_{mn}(y) = \frac{\partial x^r}{\partial y^m} \cdot \frac{\partial x^s}{\partial y^n} \cdot T_{rs}(x) = \frac{\partial x^r}{\partial y^m} \cdot \frac{\partial x^s}{\partial y^n} \cdot \frac{\partial V_r(x)}{\partial x^s}$$

$$\frac{\partial V_m(y)}{\partial y^n} = \frac{\partial}{\partial y^n}\left[\frac{\partial x^r}{\partial y^m} \cdot V_r(x)\right] \neq \frac{\partial x^r}{\partial y^m} \cdot \frac{\partial V_r(x)}{\partial y^n}$$

Ordinary differentiation of a tensor is not correct because the differential of a product is a sum of two terms. The extra term would be zero if the coordinates did not vary in direction from place to place in the space such as in Cartesian coordinates. Therefore, we need to do something different to differentiate tensors. We need to add a term that takes out the second term, which is the variation of coordinates.

$$\nabla_p V_n \equiv \frac{\partial V_m}{\partial y^p} + \Gamma^r_{pm} V_r \qquad \text{wherein} \qquad \Gamma^r_{mn} \equiv \frac{\partial}{\partial y^n}\left(\frac{\partial x^r}{\partial y^m}\right)$$

∇ is the covariant derivative, and Γ is called a Christofel symbol. The covariant derivative of a scalar is the ordinary derivative. The pattern is as given below. For each index of the tensor, there is a term with Γ. For each index get an additional term turning a "blind eye" to all others.

$$\nabla_p T_{mn} = \frac{\partial T_{mn}}{\partial y^p} + \Gamma^r_{pm} T_{rn} + \Gamma^r_{pn} T_{mr}$$

Γ is constructed from components of the Metric tensor. One property of a Metric tensor is that its covariant derivative is zero. We can derive Γ from this statement.

$$\nabla_p g_{mn} = \frac{\partial g_{mn}}{\partial y^p} + \Gamma^s_{pm} g_{sn} + \Gamma^r_{pn} g_{mr} = 0$$

$$\Gamma^a_{bc} = \frac{1}{2} \cdot g^{ad}_{(y)} \cdot \left\{\frac{\partial g^{(y)}_{dc}}{\partial y^b} + \frac{\partial g^{(y)}_{db}}{\partial y^c} - \frac{\partial g^{(y)}_{bc}}{\partial y^d}\right\} = \frac{1}{2} \cdot g^{ad}_{(y)} \cdot \Gamma_{bcd}$$

$$\nabla_n V_m = \frac{\partial V_m}{\partial x^n} + \Gamma^r_{mn} \cdot V_r$$

We can differentiate objects with upper or lower indices.

$$\nabla_m V^n = \frac{\partial V^n(y)}{\partial y^m} + \Gamma^n_{mr} \cdot V^r$$

Suppose we have a vector field $V_n(y)$ (not in Cartesian coordinates). How does any vector vary along a curve?

$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial y^m} \cdot \frac{dy^m}{ds}$ wherein $\frac{dy^m}{ds}$ is a unit tangent vector.

$$\nabla_s V^n(y) = \nabla_m V^n(y) \cdot \frac{dy^m}{ds} = \frac{\partial V^n}{\partial y^m} \cdot \frac{dy^m}{ds} + \Gamma^n_{mr} V^r \cdot \frac{dy^m}{ds}$$

How do tangent vectors vary along the curve? The covariant derivative of a tangent vector is a tensor as follows.

$$= \frac{d^2 y^n}{ds^2} + \Gamma \cdot \frac{dy^n}{ds} \cdot \frac{dy^r}{ds}$$

A Geodesic curve is defined as that curve for which ∇'s of its tangent vectors are zero. A Geodesic is the straightest possible curve of the space. Particles in space move along Geodesics.

In classical mechanics $m \cdot \frac{d^2 y}{dx^2} + m \cdot \frac{\partial V}{\partial y}$, wherein V is a scalar gradient.

In general relativity this equation becomes the following.

$$\frac{d^2 y^n(s)}{ds^2} + \Gamma^n_{mr} \cdot \frac{dy^m}{ds} \cdot \frac{dy^r}{ds} = 0$$

End lecture #5