Einstein's Theory of Relativity, PHY 27
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Session 7, November 3, 2008

Summary of Concepts

More on parallel transport
Riemann Curvature Tensor
Ricci Tensor
Flow of Energy and Momentum

Parallel transport of a vector around a closed path that surrounds an area that contains a point of curved space will result in the vector returning to its starting point with a non-zero deflection with respect to its orientation at the start of the move. One could use a gyroscope to achieve such parallel transport. Parallel transport changes the angle (θ) and not the length. The curvature R of the space that results in such a deflection is given by;

$R = \frac{d\theta}{da}$ where a is the area contained within the loop.

If we transport CCW about the closed path, and the vector deflects CCW, then the curvature of the space is defined as positive.

To determine deflection of a vector we must define a reference. Two directions define a plane in multidimensional space. In general we must defined two planes by which to measure parallel transport; one plane in which we move along the closed path, and the other in which the vector moves (deflects). Let δV^{μ} be the deflection of the vector.

$$\delta V^{\mu} = R^{\mu}_{\sigma\upsilon\tau} dx^{\upsilon} \cdot dx^{\tau} \cdot V^{\sigma}$$

Set the covariant differential of the vector to zero.

$$\frac{dV^{\mu}}{ds} + \Gamma^{\mu}_{\alpha\beta} V^{\alpha} \cdot \frac{dx^{\beta}}{ds} = 0 \quad \text{or} \quad dV = -\Gamma^{\mu}_{\alpha\beta} V^{\alpha} \cdot dx^{\beta}$$

Γ is not equal to zero, but it varies as the vector moves around the loop. Therefore, R is a function of the derivative of Γ.

$$R^{\rho}_{\sigma\mu\upsilon} = \partial_{\mu}\Gamma^{\rho}_{\upsilon\sigma} - \partial_{\upsilon}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\cdot\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\rho}_{\upsilon\lambda}\cdot\Gamma^{\lambda}_{\mu\sigma} \quad \text{where} \quad \partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}$$

R is the Riemann Curvature Tensor, and it always has four indices. Two indices each are for the two planes that must be tracked during the move around the loop. The indices υ and σ define the plane of motion in space and the other two (ρ and μ) define how the vector changes in a plane. If we lower the index ρ, we can better see the symmetries of *R*.

$$R_{\lambda\sigma\mu\upsilon} = g_{\lambda\sigma}\cdot R^{\rho}_{\sigma\mu\upsilon} \quad \text{and} \quad R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}$$

For two dimensions there is one component, for three dimensions there are three components and for four dimensions there are 24 components.

The numerical curvature of the Earth's surface is the inverse of the square of the radius of the Earth. The Ricci tensor is defined as a contraction of indices of the curvature tensor.

$$R_{\delta\beta} = R_{\beta\alpha} \equiv R_{\alpha\beta\gamma\delta}\cdot g^{\alpha\gamma}$$

The theory of relativity is one-half geometry and one-half mass. Mass alters geometry and geometry is the way mass moves.

Let us now think of flow in space. Consider a box in space (not space-time). If electric charges are moving through the box, how much charge is in the box at any given time? In terms of space, the current density is

$$\frac{d^2Q}{dA_x\cdot dt} = J^x$$ where A_x is cross-sectional area of the box normal to x-axis.

In space-time, the charge current becomes a four-vector (recall c =1 to make the units correct to convert time to distance).

$$J^{\mu} = \left(\rho, J^x, J^y, J^z\right)$$

The Continuity equation expresses the conservation of charge (charge changes only by flowing through the walls of the box).

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = \frac{\partial\rho}{\partial t} + \frac{\partial J^x}{\partial x} + \frac{\partial J^y}{\partial y} + \frac{\partial J^z}{\partial z} = 0$$

$$\frac{\partial J^0}{\partial x^0}+\frac{\partial J^1}{\partial x^1}+\frac{\partial J^2}{\partial x^2}+\frac{\partial J^3}{\partial x^3}=\frac{\partial J^\mu}{\partial x^\mu}=0$$

While charge is a scalar (invariant), energy and momentum are not invariant.

$$(E,P)=\left(P^0,P^x,P^y,P^z\right)$$

Where E is the time component and P are the space components. The four-vector is conserved because each component is conserved separately. Now consider the notion of flow as above.

$$T^{\mu\upsilon}\equiv\begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}\begin{matrix} \leftarrow \textit{energy density} \\ \leftarrow \textit{momentum density} \\ \leftarrow \textit{momentum density} \\ \leftarrow \textit{momentum density} \end{matrix}$$

If each component of T is conserved, then can write a continuity equation.

$$\left\{\begin{matrix} \frac{\partial T^{00}}{\partial t}+\frac{\partial T^{01}}{\partial x}+\frac{\partial T^{02}}{\partial y}+\frac{\partial T^{03}}{\partial z} \\ \frac{\partial T^{10}}{\partial t}+\frac{\partial T^{11}}{\partial x}+\frac{\partial T^{12}}{\partial y}+\frac{\partial T^{13}}{\partial z} \\ \frac{\partial T^{20}}{\partial t}+\frac{\partial T^{21}}{\partial x}+\frac{\partial T^{22}}{\partial y}+\frac{\partial T^{23}}{\partial z} \\ \frac{\partial T^{30}}{\partial t}+\frac{\partial T^{31}}{\partial x}+\frac{\partial T^{32}}{\partial y}+\frac{\partial T^{33}}{\partial z} \end{matrix}\right\}=\frac{\partial T^{\upsilon\mu}}{\partial x^\mu}=0$$

The curvature of space is caused by energy and momentum (recall that mass is equivalent energy from $E=M\cdot C^2$). The motions of particles in a gravitation field with no other applied forces move along geodesic curves.

$$\frac{d}{d\tau}\left(\frac{dx^\mu}{d\tau}\right)+\Gamma^\mu_{\mu\sigma}\cdot\frac{dx^\upsilon}{d\tau}\cdot\frac{dx^\sigma}{d\tau}=0$$

Compare this expression with Newtonian motion?

$$m \cdot \frac{d}{dt}\left(\frac{dx}{dt}\right) + F_g = 0$$

So, the Christofel symbol is related to gravity Γ^{μ}_{00}.

End Lecture 7