Einstein's Theory of Relativity, PHY 27
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Summary of Concepts
Operators
Commutators
Riemann curvature tensor
Ricci Tensor
Curvature Scalar
Newtonian Approximation

Operators

We can view curvature as a commutator. Linear operator on vector space. Suppose we have a collection of functions as components of a vector.

Examples of operators

Derivative operator $\frac{\partial}{\partial x^\mu}$

Multiply operator $f(x)\bullet$

Scalar Matrix operator $M^\alpha_\beta \cdot V^\beta \equiv V^\alpha$

Functional Matrix operator $M(x)^\alpha_\beta \cdot V(x)^\beta \equiv V^\beta(x)$

The covariant derivative is a combination of a derivative operator and a functional matrix operator

$$\nabla_\mu V^\alpha(x) = \frac{\partial V^\alpha(x)}{\partial x^\alpha} + \Gamma^\alpha_{\mu\beta}$$

$$\nabla_\mu = \frac{\partial}{\partial x^\mu} + \Gamma_\mu(x)$$

Γ_μ is a set of four matrices.

Commutators

Given two operators A and B, a commutator is defined as follows.

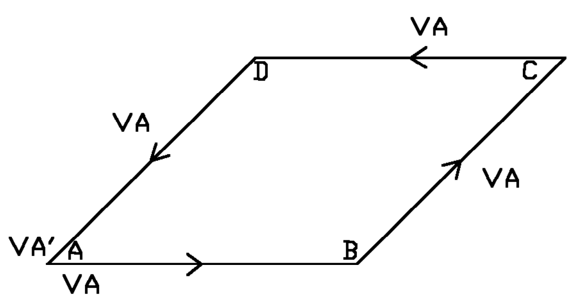
$$[A,B] \equiv AB - BA$$

$$\left[\frac{\partial}{\partial x}, f(x)\right] = \frac{\partial f(x)}{\partial x} - f(x)\cdot\frac{\partial}{\partial x} \quad \text{[Check sign?]}$$

Let this commutator act on $V(x)$

$$\frac{\partial f(x)}{\partial x}\cdot V(x) - f(x)\cdot\frac{\partial V(x)}{\partial x} = \frac{\partial f(x)}{\partial x}\cdot V(x) + \frac{\partial V(x)}{\partial x}\cdot f(x) - f(x)\cdot\frac{\partial V(x)}{\partial x} = V(x)\cdot\frac{\partial f(x)}{\partial x}$$

$$\therefore \left[\frac{\partial}{\partial x}, f(x)\right]V(x) = \frac{\partial f(x)}{\partial x}\cdot V(x) \qquad \frac{\partial f(x)}{\partial x} - f(x)\cdot\frac{\partial}{\partial x} = \frac{\partial f(x)}{\partial x}$$



$$V_A - V_{A'} = \left[(V_C - V_D) - (V_B - V_A)\right] - \left[(V_C - V_B) - (V_D - V_{A'})\right]$$

Note the right-hand side contains second differences. Now convert this equation to differentials.

$$\nabla V = \delta x^\upsilon \cdot dx^\mu \cdot \nabla_\upsilon \nabla_\mu V - \delta x^\upsilon \cdot dx^\mu \cdot \nabla_\mu \nabla_\upsilon V$$

Around a closed loop

$\delta V = dx^{\mu} \cdot \delta x^{\upsilon} \cdot \left[\nabla_{\upsilon}, \nabla_{\mu}\right] V$ where

$$\left[\nabla_{\upsilon}, \nabla_{\mu}\right] = (\partial_{\upsilon} + \Gamma_{\upsilon}) \cdot (\partial_{\mu} + \Gamma_{\mu}) - (\partial_{\mu} + \Gamma_{\mu}) \cdot (\partial_{\upsilon} + \Gamma_{\upsilon})$$

$$\left[\nabla_{\upsilon}, \nabla_{\mu}\right] = -\frac{\partial \Gamma_{\upsilon}}{\partial x^{\mu}} + \frac{\partial \Gamma_{\mu}}{\partial x^{\upsilon}} + \Gamma_{\upsilon} \cdot \Gamma_{\mu} - \Gamma_{\mu} \cdot \Gamma_{\upsilon}$$

Therefore,

$$\delta V^{\alpha} = \delta x^{\upsilon} \cdot dx^{\mu} \cdot \left(\frac{\partial \Gamma^{\alpha}_{\mu\beta}}{\partial x_{\upsilon}} - \frac{\partial \Gamma^{\alpha}_{\upsilon\beta}}{\partial x_{\mu}} + \Gamma^{\alpha}_{\upsilon\delta} \cdot \Gamma^{\delta}_{\beta} - \Gamma^{\alpha}_{\mu\delta} \cdot \Gamma^{\delta}_{\beta} \right) \cdot V_{\beta}$$

And the quantity in parentheses is the Riemann (curvature) tensor $(R^{\alpha}_{\upsilon\mu})$. The quantity $\delta x^{\upsilon} \cdot dx^{\mu}$ defines the plane.

$$R^{\alpha}_{\upsilon\beta\mu} = \left(\frac{\partial \Gamma^{\alpha}_{\mu\beta}}{\partial x_{\upsilon}} + \Gamma^{\alpha}_{\upsilon\delta} \cdot \Gamma^{\delta}_{\mu\beta} \right) - \left(\frac{\partial \Gamma^{\alpha}_{\upsilon\beta}}{\partial x_{\mu}} + \Gamma^{\alpha}_{\mu\delta} \cdot \Gamma^{\delta}_{\upsilon\beta} \right)$$

Use the metric tensor to lower the index. What are the symmetries?

$$R_{\upsilon\mu\alpha\beta} = R_{\beta\mu\alpha\upsilon} \qquad R_{\upsilon\mu\alpha\beta} = -R_{\mu\upsilon\alpha\beta} \qquad R_{\upsilon\mu\alpha\beta} = -R_{\upsilon\mu\beta\alpha}$$

The Ricci tensor $(R_{\mu\beta})$ is made from the Riemann tensor by using the metric tensor to contract the indices υ and α as follows.

$$g^{\upsilon\alpha} \cdot R_{\mu\upsilon\alpha\beta} = R^{\alpha}_{\mu\beta\alpha} = R_{\mu\beta} = R_{\beta\mu}$$

If you try to contract any other indices, you will get zero because of symmetry.

The curvature scalar (R) is obtained from the Ricci tensor as follows.

$$R = g^{\beta\mu} \cdot R_{\beta\mu}$$

In general a vanishing curvature scalar (everywhere in the space) is necessary, but not sufficient, to indicate flatness of the space. However, in a space of two dimensions, a vanishing curvature scalar is necessary and sufficient to indicate flatness. In a space of three dimensions the vanishing of the Ricci tensor is necessary and sufficient to indicate flatness of the space.

Two types of curvature are intrinsic and extrinsic. Given a space that can be represented by a piece of paper that is folded, intrinsic curvature is within the paper itself, while extrinsic curvature is how the paper is folded or unfolded.

The Riemann tensor is the gravitation field. Consider the Newtonian approximation, which is valid for small field strengths (i.e., not near a black hole) and slow motion (slow compared with the speed of light). The equation of motion is given by the covariant derivative of the tangent vector follows.

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\tau\lambda} \cdot \frac{dx^\tau}{d\tau} \cdot \frac{dx^\lambda}{d\tau}$$

where $d\tau^2 = g_{\mu\upsilon} \cdot dx^\upsilon \cdot dx^\mu$ and $\frac{dx^\mu}{d\tau}$ is tangent vector

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\sigma\lambda} \cdot \frac{dx^\sigma}{d\tau} \cdot \frac{dx^\lambda}{d\tau} = 0$$

$m \cdot \frac{dx^2}{dt^2} = -\Gamma^\mu_{00} \cdot \frac{dx^0}{d\tau} \cdot \frac{dx^0}{d\tau} = -\Gamma^x_{00}$ (note m is usually set to unity)

$$\Gamma_{00}^{x} = \frac{1}{2} \cdot g^{xx} \cdot \left(\frac{\partial g_{x0}}{\partial x^0} + \frac{\partial g_{0x}}{\partial x^0} - \frac{\partial g_{00}}{\partial x} \right) = \eta^{xx} = -1 = \frac{1}{2} \cdot \left(\frac{\partial g_{x0}}{\partial x^0} + \frac{\partial g_{0x}}{\partial x^0} - \frac{\partial g_{00}}{\partial x} \right)$$

where η_{xx} is Minkowski metric of special relativity.

In the Neutonian approximation the time component can be neglected, which leaves only the space component.

$$m \cdot \frac{d^2x}{dt^2} = -\frac{1}{2} \cdot \frac{\partial g_{00}}{\partial x} \quad \text{and} \quad \frac{d^2x}{dt^2} = -\frac{\partial \phi}{\partial x}$$

where ϕ is the potential gradiant.

$$\therefore \boxed{g_{00} = 2 \cdot \phi + Const} \quad \text{or} \quad \phi = \frac{1}{2} \cdot g_{00}$$

Let $\vec{A}(x)$ be the acceleration field and ρ be the mass density.

$$\vec{A}(x) = -\frac{\partial \phi}{\partial x} \qquad \nabla A = -4 \cdot \pi \cdot \rho \cdot G \qquad \nabla^2 \phi = 4 \cdot \pi \cdot \rho \cdot G = 4 \cdot \pi \cdot G \cdot T^{00}$$

$$\nabla^2 g_{00} = 8 \cdot \pi \cdot G \cdot T^{00}$$

But this is not a tensor equation because the indices are wrong. What is the tensor equation under the Neutonian approximation?

Let $G_{\mu\upsilon} = 8 \cdot \pi \cdot G \cdot T_{\mu\upsilon}$ and find $G_{\mu\upsilon}$

$$A \cdot R_{\mu\upsilon} + B \cdot g_{\mu\upsilon} = 8 \cdot \pi \cdot G \cdot T_{\mu\upsilon}$$

We will find A and B to get the Einstein Tensor.

End Lecture #8