Einstein's Theory of Relativity, PHY 27
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Summary of Concepts

Field equations of relativity
Correspondence
Einstein Tensor
Einstein Equations

$$g_{\alpha\beta} \cdot g^{\alpha\beta} = \delta_{\alpha}^{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Use g to raise or lower indices.

$$g_{\alpha\beta} \cdot T^{\alpha\gamma} = T_{\alpha}^{\gamma}$$

$g_{\alpha}^{\gamma} = \delta_{\alpha}^{\gamma}$ always

Use g for contraction of indices.

$$g_{\alpha\beta} \cdot T^{\alpha\beta} = g_{\alpha}^{\beta} \cdot T_{\beta}^{\alpha} = \delta_{\alpha}^{\beta} \cdot T_{\beta}^{\alpha} = T_{\alpha}^{\alpha} = T_{11} + T_{22} + \ldots = scalar$$

In ordinary space, g is positive definite, i.e., all its eigne values are positive. In Special relativity

$$d\tau^2 = dt^2 - \left(\frac{dx^2 + dy^2 + dz^2}{c^2} \right)$$

So $d\tau$ can be negative, which means it cannot be positive definite.

$g_{mn} \cdot dx^m \cdot dx^n$ can be positive, zero or negative. Although in this class I prefer to define proper space time as above, you will often see it as

$d\tau^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$

One can write it either way, but one must be consistent.

The Metric tensor is smooth, which means it is differentiable, symmetric (real eigen values) and invertable (no zero eigen values).

The field equations of Relativity

$g_{11} = g_{22} = g_{33} = -1 + small$ and $g_{12} ... = 0 + small$

We require correspondence with Newtonian approximation.

$$Energy = m \cdot c^2 + \frac{1}{2} \cdot m \cdot v^2 \qquad Momentum = m \cdot v$$

The momenta are much smaller than energy because of the size of the speed of light. For objects are rest or moving much more slowly than the speed of light;

$$T^{\mu\upsilon} = \begin{pmatrix} T^{00} & 0 \\ 0 & 0 \end{pmatrix}$$

The inverse-square-law rule is expressed as follows. ϕ is gravitational potential.

$$\nabla^2\phi = 4\cdot\pi\cdot G\cdot\rho \qquad a = -\nabla\phi \qquad a_m = -\partial_m\phi$$

$$\phi = \frac{1}{2}\cdot g_{00} + Constant \qquad \text{(sign???)}$$

Einstein tried the following equation, but he ran into a problem with conservation of energy and momentum.

$$R^{\mu\upsilon} = k\cdot T^{\mu\upsilon} \qquad \text{(not correct)}$$

Continuity equation $\partial_\mu J^\mu = 0$ try $\partial_\varpi T^{\mu\upsilon} = 0$ but must do a covariant derivative.

$$\nabla_\mu T^{\mu\upsilon} = 0 \quad \text{correct equation}$$

if $R^{\mu\upsilon} = k\cdot T^{\mu\upsilon}$ then $\phi = \nabla_\mu R^{\mu\upsilon} = k\cdot\nabla_\mu T^{\mu\upsilon}$

The above is not true if tensor is the Ricci tensor.

$$\nabla_\mu R^{\mu\upsilon} = \frac{1}{2}\cdot g^{\mu\upsilon}\cdot\partial_\mu R$$

Where $R^{\mu\upsilon}$ is the Ricci Curvature Tensor and R is the Ricci Curvature scalar. Can do this because R is a scalar.

$$\nabla_\mu R^{\mu\upsilon} = \frac{1}{2}\cdot g^{\mu\upsilon}\cdot\nabla_\mu R = \frac{1}{2}\cdot\nabla_\mu g^{\mu\upsilon}\cdot R \quad \text{since}$$

$$\nabla_\mu\left(A\cdot B\right) = A\cdot\nabla B + \left(\nabla A\right)\cdot B \qquad \left(\partial g^{\mu\upsilon} = 0\right)$$

The following was a big breakthrough for Einstein. Let *G* be the Einstein Tensor. The following is a set of 4 continuity equations.

$$\nabla_{\mu} G^{\mu\upsilon} \equiv \nabla_{\mu}\left[R^{\mu\upsilon} - \frac{1}{2}\cdot g^{\mu\upsilon}\cdot R\right] = 0$$

$R^{\mu\upsilon} - \frac{1}{2}\cdot g^{\mu\upsilon}\cdot R = k\cdot T^{\mu\upsilon}$ This comprises 16 equations.

If $T^{\mu\upsilon} = 0$ then $g_{\mu\upsilon}\cdot\left(R^{\mu\upsilon} - \frac{1}{2}\cdot g^{\mu\upsilon}\cdot R\right) = 0$

$R - \frac{4}{2}\cdot R = 0 \qquad R = 0$

If the Curvature Scalar equals zero, then energy-momentum density is zero. But this does not mean space is flat, because there may be gravity waves.

Note $g_{\varepsilon\upsilon}\cdot g^{\upsilon\mu} = \delta^{\mu}_{\alpha} \qquad g_{\mu\upsilon}\cdot g^{\mu\upsilon} = \delta^{\mu}_{\mu} = 4$

$k\cdot T^{\mu\upsilon}$ is not the total energy and momentum in space. Gravity waves carry energy.

In 3-space with no gravitational waves, Ricci tensor equals zero is complete.

$$R^{\mu\upsilon} - \frac{1}{2}\cdot g^{\mu\upsilon}\cdot\nabla_{\mu} R = k\cdot T^{\mu\upsilon}$$

Contract indices by means of $g^{\mu\upsilon}$

$$-R = k\cdot g_{\mu\upsilon}\cdot T^{\mu\upsilon} = k\cdot T^{\mu}_{\mu} = k\cdot T$$

Substitute into above

$$R^{\mu\upsilon} + \frac{k}{2} \cdot g^{\mu\upsilon} \cdot T = k \cdot T^{\mu\upsilon}$$

To obtain another form of Einstein's equation

$$R^{\mu\upsilon} = k \cdot \left(T^{\mu\upsilon} - \frac{1}{2} \cdot g^{\mu\upsilon} \cdot T \right)$$

End Lecture 9