(1) $d\phi(x_1 \cdots x^{\cdot}) = \frac{\partial \phi}{\partial x^1} dx^1 + \frac{\partial \varphi}{\partial x^2} dx^2 + \cdots$

$$= \sum \frac{\partial \phi}{\partial x^m} dx^m$$

$$= \frac{\partial \phi}{\partial x^m} dx^m$$

$$= \frac{\partial \phi}{\partial x^r} dx^r$$

2) Let y Be functions of x

$y^1(x^1 \cdots x^D)$, $y^2(x^1 \cdots x^D)$

$y^m(x)$

$$\frac{\partial \phi}{\partial y^m} = \frac{\partial \phi}{\partial x^1} \frac{\partial x^1}{\partial y^m} + \frac{\partial \phi}{\partial x^2} \frac{\partial x^2}{\partial y^m}, \ldots$$

$$= \frac{\partial \phi}{\partial x^r} \frac{\partial x^r}{\partial y^m}$$

$$\frac{\partial x^m}{\partial x^n} = \frac{\partial x^m}{\partial y^r} \frac{\partial y^r}{\partial x^n} = \delta^m_n$$

NOTATIONS

n is an index that labels direction. In a D-dimensional space n runs from 1 to D.

In 2D n labels x, y or X^1, X^2

3D xyz or X^1, X^2, X^3

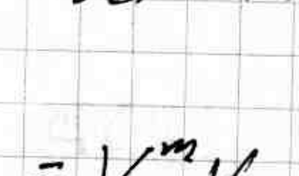
IN SPACE-TIME t, x, y, z or X^0, X^1, X^2, X^3

Summation Convention

$$V^n V_n = V^1 V_1 + V^2 V_2 + V^3 V_3 \cdots$$
$$= \sum V^n V_n$$

~~EXAMPLE~~

EXAMPLE

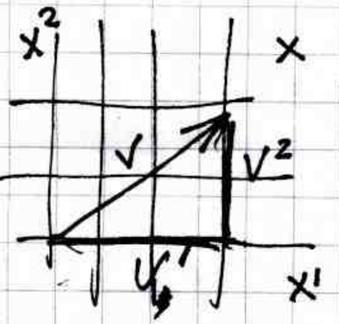


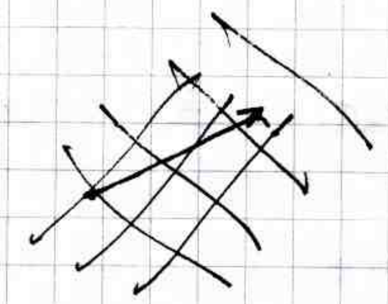
$$(\Delta S)^2 = \Delta X^m \Delta X^m$$

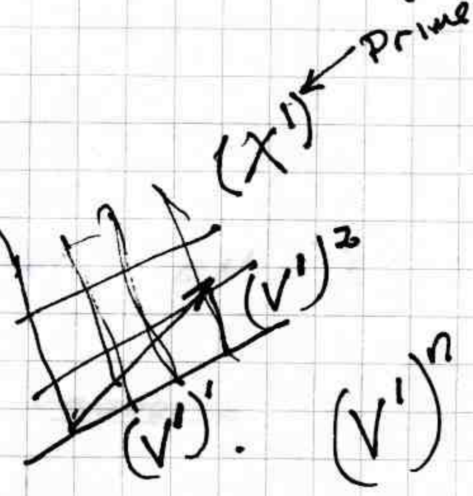
A SUMMATION INDEX MAY BE REPLACED BY ANOTHER LETTER

$$V^n V_n = V^m V_m$$

COORDINATES



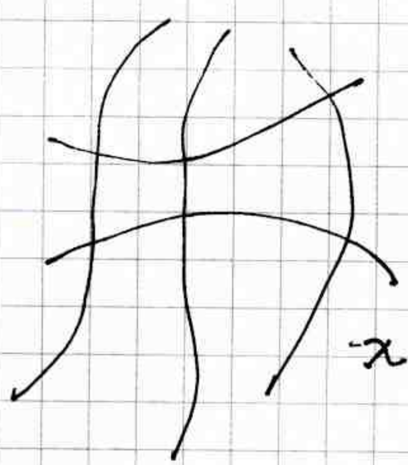


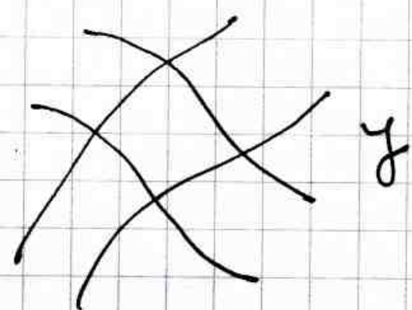


$(V')^n$

or y^m

Curvilinear Coordinates





$$y^m = y^m(x)$$

~~$\vec{dx}$ IS A~~ Φ IS A SCALAR

$$\Phi'(y) = \Phi(y)$$

VECTORS

$\vec{dx}$ = VECTOR

dx^m = contravariant components

Variations

SUPPOSE Z DEPENDS ON $X^1, \cdots X^D$

~~AND SUPPOSE~~

$$dZ = \frac{\partial Z}{\partial x^1} dx^1 + \frac{\partial Z}{\partial x^2} dX^2 + \cdots$$

$$dy^n = \frac{\partial y^n}{\partial x^r} dx^r = \frac{\partial y^n}{\partial x^1} dx^1 + \frac{\partial y^n}{\partial x^2} dx^2 \cdots$$

CONTRAVARIANT TRANSFORMATION RULE

$$V_{(y)}^n = \frac{\partial y^n}{\partial x^r} V_{(x)}^r$$

TENSORS WITH 2 COMPONENTS

A^m, B^m ARE BOTH VECTORS, EACH HAS D COMP

$T^{mn} = A^m B^n$ IS a RANK 2 TENSOR WITH D^2 COMPONENTS

$$A_{(y)}^m = \frac{\partial y^m}{\partial x^r} A_{(x)}^r$$

$$B_{(y)}^n = \frac{\partial y^n}{\partial x^s} B_{(x)}^s$$

$$T_{(y)}^{mn} = \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} A_{(x)}^r B_{(x)}^s = \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} T_{(x)}^{rs}$$

COVARIANT (DOWNSTAIRS)

VECTORS

ϕ IS A SCALAR

$\frac{\partial \varphi}{\partial x^n}$ IS A COVARIANT VECTOR.

$$\frac{\partial \varphi}{\partial y^n} = \frac{\partial \varphi}{\partial x^m} \frac{\partial x^m}{\partial y^n}$$

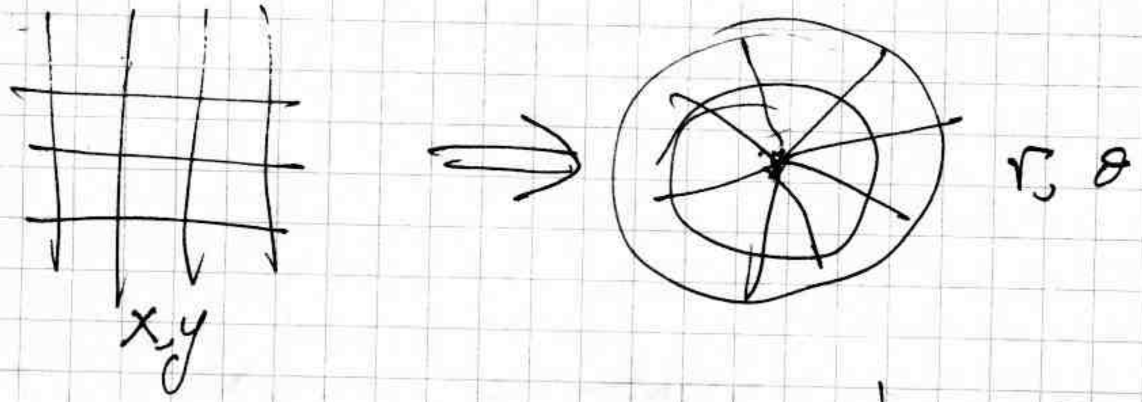
$$\frac{\partial \phi}{\partial y^1} = \frac{\partial \phi}{\partial x^1} \frac{\partial x^1}{\partial y^1} + \frac{\partial \varphi}{\partial x^2} \frac{\partial x^2}{\partial y^1} + \cdots$$

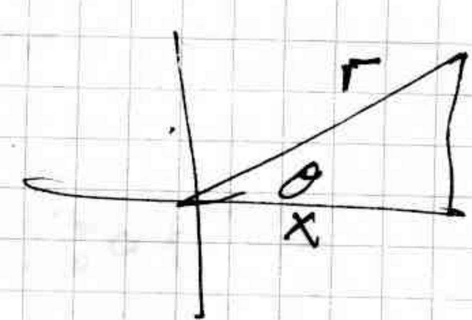
$$A_n^{(y)} = \frac{\partial x^m}{\partial y^n} A_m \qquad \text{Covariant}$$

$$A^n = \frac{\partial y^n}{\partial x^m} A^m \qquad \text{contravariant}$$

$$T_{mn}^{(y)} = \frac{\partial x^r}{\partial y^m} \frac{\partial x^s}{\partial y^n} T_{rs}^{(x)}$$

EXAMPLE: POLAR COORDS





$$x = r\cos\theta$$
$$y = r\sin\theta$$

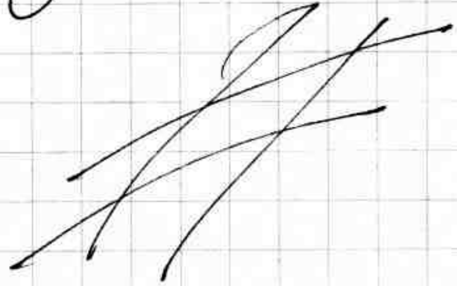
$$dx = \cos\theta\, dr - r\sin\theta\, d\theta$$
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

$$dx^2 + dy^2 = (c^2+s^2)\,dr^2 + r^2(c^2+s^2)\,d\theta^2$$
$$\cancel{-\, csr\, dr\, d\theta + scr\, dr\, d\theta}$$

$$\boxed{\begin{aligned} g_{rr} &= 1 \\ g_{r\theta} &= 0 \\ g_{\theta\theta} &= r^2 \end{aligned}}$$

OFF DIAGONAL ELEMENTS

OF g_{mn} MEAN COORDS NOT ORTHOGONAL



Is g_{mn} a tensor?

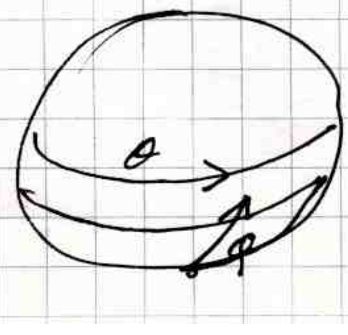
SUPPOSE $ds^2 = g^{(x)}_{mn}\, dx^m dx^n$

$$= g^{(x)}_{mn} \frac{\partial x^m}{\partial y^r} \frac{\partial x^n}{\partial y^s}\, dy^r dy^s$$

$$= g^{(y)}_{rs}\, dy^r dy^s$$

YES: g_{mn} IS A TENSOR. $\boxed{\text{SPECIAL}}$

A ~~SPACE~~ SPHERE



ϕ = POLAR ANGLE

θ = ~~LAT~~ LONGITUDE

$$ds^2 = (d\phi^2 + \sin^2\phi\, d\theta^2) R^2$$